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Rings Which Are Sums of Finite Fields

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We describe all semigroup rings and band-graded rings which are direct sums of finite fields. We show that the class of rings which are direct sums of finite fields is closed for taking sums of two subrings. © 1999 Academic Press

Semigroup rings of arithmetical semigroups have nice applications in analytic number theory (see, for example, [14, Sect. 2.1]). We shall describe all semigroup rings which are direct sums of finite fields (Theorem 3). To this end we consider another important construction (Theorem 2) and establish an interesting ring-theoretic property of the class of rings which are direct sums of finite fields (Theorem 1).

The preservation of ring properties by sums of rings has been actively investigated by many authors. Let $\mathscr{K}$ be a class of rings, and let a ring R be a sum of its subrings R_i, where $i = 1, \ldots, n$. Suppose that all the R_i are in $\mathscr{K}$. Does it follow that R belongs to $\mathscr{K}$? This question has been considered for several classes $\mathscr{K}$; see [1, 2, 6, 8] for references. We only mention that a long-standing difficult problem in this area has been recently solved in [9] and [12]. Namely, examples of rings which are not nil but are sums of two locally nilpotent subrings were constructed. Taking a homomorphic image of a ring introduced in [9], a simpler version of the example was given in [12]. Moreover, it was shown that there exists a primitive ring which is a sum of two Wedderburn radical subrings, which answered several questions known

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in the literature. A construction similar to the one introduced in [9] was used again in [13] to answer another question asked by Kegel in 1964.

In this paper we shall deal with commutative rings which are direct sums of finite fields. This class is natural: a finite commutative ring is a direct sum of finite fields if and only if it is semisimple [17, Thm. 8.4].

All rings considered will be commutative. We use the standard notation and terminology of the theory of finite fields and rings; see [15–17, 19].

THEOREM 1. *Let $\mathscr{K}$ be the class of rings which are sums of finite fields. If a finite commutative ring R is a sum of its subrings $R_i \in \mathscr{K}$, where $i = 1, \dots, n$, then $R \in \mathscr{K}$, too.*

Several authors have imposed additional restrictions on the ring R which is a sum of its subrings R_i, $i \in I$, by requiring that the indexing set I be a semilattice and R be a graded ring. Recall that a *semilattice* is a commutative semigroup entirely consisting of idempotents. Let S be a semilattice. A ring R is called an *S-graded ring* if $R = \bigoplus_{s \in S} R_s$ is an additive direct sum of subrings R_s, and $R_s R_t \subseteq R_{st}$ for all $s, t \in S$. Semilattice-graded rings have been considered in [3, 18, 20, 21] and other papers.

The problem of investigating semilattice-graded rings $F = \bigoplus_{s \in S} F_s$, where all components F_s are fields, was posed in [21]. Suppose that all the F_s are fields with identities 1_s. Then it is easily seen that the set

$$Y = \{0\} \cup \{1_s \mid s \in S\}$$

is also a semilattice, and we may regard F as a Y-graded ring, if we put $F_0 = 0$, and $F_{1s} = F_s$, for all $s \in S$. After that F satisfies one additional condition: the product of the identities of F_s and F_t is the identity of F_{st}, if $F_{st} \neq 0$. Following [18], we say that a semilattice-graded ring $F = \bigoplus_{s \in S} F_s$ is a *special S-graded ring* if all the F_s have identities 1_s, whenever $s \neq 0$, and $1_s 1_t = 1_{st}$ for all $s, t \in S$.

THEOREM 2. *Let S be a semilattice, and let $F = \bigoplus_{s \in S} F_s$ be a special S-graded ring. Then the following conditions are equivalent*:

(i) *F is a direct sum of finite fields*;

(ii) *all nonzero rings F_s are direct sums of finite fields and every principal ideal of S is finite.*

Next, we apply Theorem 2 to semigroup rings. A semigroup S is said to be *torsion-disjoint* with a ring R, if $s^n = t^n$ and $mr = 0$ imply $s = t$ or $r = 0$, whenever $s, t \in S$, $r \in R$, m, n positive integers, and m divides n.

THEOREM 3. *A nonzero semigroup ring RS is a direct sum of finite fields if and only if R is a direct sum of finite fields, S is a commutative semigroup torsion-disjoint with R, and every principal ideal of S is finite.*

For our proof we need the following lemma.

LEMMA 4. *In a commutative ring the product of two finite fields is either zero or a finite direct sum of finite fields.*

Proof. Let R be a commutative ring, and let A and B be finite fields contained in R. If they have different characteristics, then $AB = 0$. Assume that they both have characteristic p. If the product of their identities is zero, then $AB = 0$. Suppose that the product of their identities is nonzero. Then AB is a ring with identity.

Suppose that A satisfies $x^{p^m} = x$ and B satisfies $x^{p^n} = x$. Take any element in AB, say $r = \sum_{i=1}^{k} a_i b_i$, where $a_i \in A$, $b_i \in B$. Put $N = p^{mn}$. Clearly, AB has characteristic p, and so it satisfies $(x+y)^p = x^p + y^p$, whence it satisfies $(x+y)^N = x^N + y^N$. Therefore $r^N = \sum_{i=1}^{k} (a_i b_i)^N$. Further, for any $0 \le i \le k$, we get $a_i^N = (a_i^{p^{mn}}) = (a_i^{p^m})^{p^{m(n-1)}} = a_i^{p^{m(n-1)}}$, because A satisfies $x^{p^m} = x$. Continuing this, we get $a_i^N = a_i^{p^{m(n-2)}} = \cdots = a_i^{p^m} = a_i$. Similarly, $b_i^N = b_i$. Hence $r^N = r$.

Therefore AB has no nonzero nilpotent elements. Since it is finite, it follows that it is a direct sum of finite fields. ■

Proof of Theorem 1. It follows from Lemma 4 that, for any $1 \le i, j \le n$, the ring $R_i R_j$ is a direct sum of finite fields. Therefore $R_{i_1} R_{i_2} \cdots R_{i_k}$ is a direct sum of finite fields, for every $1 \le i_1, \ldots, i_k \le n$.

Let $k \le n$ be the maximum positive integer such that there exists a nonzero product $R_{i_1} \cdots R_{i_k}$, where all $i_1, \ldots, i_n$ are pairwise distinct. We shall prove our theorem by induction on k.

If $k = 1$, then all the products $R_i R_j$ are equal to zero for all $1 \le i \ne j \le n$. Therefore R is a direct sum of $R_1, \ldots, R_n$, and the conclusion follows.

Suppose that $m \le n - 1$ and that for all rings R with values of k not exceeding m our theorem has been proved. Consider a ring R with $k = m + 1$.

As noted above, the nonzero product $P = R_{i_1} \cdots R_{i_{m+1}}$ is a direct sum of fields, and so it has an identity e. Therefore R is a direct sum of P and $(1 - e)R$. Hence we may factor P out and assume that $P = 0$. Similarly, if there are other products of $m + 1$ factors which are nonzero, they will be direct summands, too, and we will be able to factor them out as well. Thus we get a ring which has $k \le m$. The induction assumption completes our proof. ■

The following example shows that our proof of Theorem 1 cannot be simplified by omitting Lemma 4.

EXAMPLE 5. The product of two finite fields in a commutative ring may not be not a field. Indeed, take a finite field $F = GF(p^n)$, where $n > 1$, with the Frobenius automorphism θ (see [16, Sect. 13.12]). The direct sum $D = F \times F$ has subfields $E = \{(x, x) | x \in F\}$ and $S = \{(x, \theta(x)) | x \in F\}$. It is routine to verify that $D = ES$.

For infinite fields the assertion analogous to Lemma 4 is not valid. Indeed, let F be a field. In the ring of rational functions

$$F(x, y) = \{ f(x, y)/g(x, y) | f, g \in F[x, y], g \neq 0\}$$

we consider the set R of all fractions of the form $f(x, y)/g(x)h(y)$. Clearly, R is a subring of $F(x, y)$. Let I be the ideal generated by $(x+y)^2$ in R. Then $I \cap F[x] = 0$. Indeed, if $(x+y)^2 f(x, y) = g(x)h(y)$, then we can substitute $y = -x$, and we get $0 = g(x)h(-x)$. Since $F[x]$ is a domain, it follows that $h(-x) = 0$, and so $h(x) = 0$, a contradiction. Similarly, $I \cap F[y] = 0$. Therefore R/I is the product of two subfields $F(x)$ and $F(y)$. However, R/I has a nilpotent element $x + y$. It is nonzero. Indeed, if $(x+y)g(x)h(x) = (x+y)^2 f(x, y)$, then we get a contradiction with the fact that $F[x, y]$ is a unique factorization domain. Thus R/I is not a direct sum of fields.

Proof of Theorem 2. Denote by 1_s the identity of F_s. Recall that every semilattice is a partially ordered set with respect to the natural partial order $e \leq f \Leftrightarrow ef = e$ (see [7]).

The "if" part: Suppose that all the F_s are direct sums of finite fields and that every principal ideal of S is finite. For any $e \in S$, denote by M_e the finite set of all maximal elements in $eS \setminus \{e\}$. Define an element $\bar{e}$ in F putting $\bar{e} = \prod_{f \in M_e} (1_e - 1_f)$. If $M_e = \varnothing$, then we put $\bar{e} = 1_e$. Since $M_e \subseteq eS$, we get $f < e$ for all $f \in M_e$, whence $1_e - 1_f$ is an idempotent, and so $\bar{e}$ is an idempotent, too.

Take any element $x = \sum_{s \in S} x_s \in F$, where $x_s \in F_s$, and put $\text{supp}(x) = \{s \in S | x_s \neq 0\}$. We shall show by induction on the cardinality of the finite set $S\,\text{supp}(x)$ that x belongs to the sum $\sum_{e \in S} F\bar{e}$.

If $|S\,\text{supp}(x)| = 1$, then $x = x_e$ for some $e \in S$, and so $\text{supp}(x) = \{e\}$. Therefore $|S\,\text{supp}(x)| = 1$ implies $|eS| = 1$, whence $eS = \{e\}$. It follows that $M_e = \varnothing$, $\bar{e} = 1_e$, and $x \in F_e = F\bar{e}$.

Suppose that $|S\,\text{supp}(x)| = n > 1$. Choose a maximal element m in $S\,\text{supp}(x)$, and put $Y = S\,\text{supp}(x) \setminus \{m\}$.Put $y = \sum_{y \in Y} x_y$. Then $x = x_m + y$. Letting $z = x - x_m \bar{m}$ we get $x = x_m \bar{m} + z$. Evidently, $\text{supp}(z) \subseteq Y$. Since $|Y| < n$, by the induction assumption $z \in \sum_{e \in S} F\bar{e}$. Hence x belongs to this sum, too. Thus $F = \sum_{e \in S} F\bar{e}$.

For any $e \neq g \in S$, either $e \not< g$ or $g \not< e$. Suppose that $e \not< g$. Then $e \neq eg$. Given that the ideal eS is finite, there exists $f \in M_e$ such that $f \geq eg$. It follows that $(1_e - 1_f)1_g = 0$. Hence $\bar{e}1_g = 0$, and so $\overline{eg} = \bar{e}1_g\bar{g} = 0$. Therefore $(F\bar{e})(F\bar{g}) = 0$. It follows that $F = \bigoplus_{e \in S} F\bar{e}$ is a direct sum of the rings $F\bar{e}$.

Clearly, every ring $F\bar{e}$ is isomorphic to F_e, and so it is a direct sum of finite fields. Therefore F is a direct sum of finite fields, too.

The "only if" part: Suppose that F is a direct sum of finite fields. Then every principal ideal of F is finite. Lemma 6 of [11] tells us that every ring F_s is a homomorphic image of F, and so all the F_s are direct sums of finite fields.

Suppose to the contrary that a principal ideal eS of S is infinite. The last exercise in [5, Sect. 1.1] tells us that $eS \setminus \{e\}$ contains an infinite descending chain, an infinite ascending chain, or an infinite set of incomparable elements.

If $eS \setminus \{e\}$ contains an infinite descending chain $e_1 > e_2 > \cdots$, then the principal ideal generated in F by 1_{e_1} is infinite, because it contains all $1_{e_2}, 1_{e_3}, \ldots$, a contradiction.

If $eS \setminus \{e\}$ has an infinite ascending chain $e_1 < e_2 < \cdots$, then F contains an infinite descending chain of idempotents $1_e - 1_{e_1} > 1_e - 1_{e_2} > \cdots$, and so 1_e generates an infinite ideal in F, a contradiction again.

If $eS \setminus \{e\}$ contains an infinite set of incomparable idempotents $e_1, e_2, \ldots$, then it is easily seen that F has a descending chain of idempotents

$$(1_e - 1_{e_1}) > (1_e - 1_{e_1})(1_e - 1_{e_2}) > \cdots > \prod_{i=1}^{n} (1_e - 1_{e_i}) > \cdots.$$

Therefore 1_e generates an infinite ideal in F. This contradiction completes the proof. ∎

Proof of Theorem 3. The "if" part: Suppose that every principal ideal of S is finite, S is torsion-disjoint with R, and R is a direct sum of finite fields R_i, $i \in I$. We see that S is torsion-disjoint with all the R_i. Clearly, it suffices to show that each R_iS is a direct sum of finite fields. Therefore to complete the proof we may assume that R is a finite field.

If a semigroup is torsion-disjoint with any ring which has nonzero periodic elements, then S is *separative*, i.e., $s^2 = st = t^2$ implies $s = t$ for any $s, t \in S$. Every periodic separative semigroup is a semilattice of groups (see [4. Sect. 4.3]). This means that there exists a semilattice Y and subgroups G_y, $y \in Y$, of S such that $S = \bigcup_{y \in Y} G_y$ is a disjoint union, and $G_xG_y \subseteq G_{xy}$ for all $x, y \in Y$.

If e_y stands for the identity of G_y, then it is easily seen that $e_xe_y = e_{xy}$. Denote the identity of R by 1. Then $1e_y$ is the identity of RG_y. In addition, $(1e_x)(1e_y) = 1e_{xy}$. Thus $RS = \bigoplus_{y \in Y} RG_y$ is a special semilattice-graded ring.

Since all principal ideals of S are finite, it follows that all principal ideals of Y are finite, too, and that all the groups G_y are finite. By [10, Thm. 2.4], all finite rings RG_y are semisimple. Therefore they are all direct sums of finite fields. Hence Theorem 2 shows that RS is a direct sum of finite fields.

The "only if" part: Suppose RS is a direct sum of finite fields. Since R is a homomorphic image of RS, we see that R is a direct sum of finite fields R_i, $i \in I$.

Given that every principal ideal of RS is finite, it follows that every principal ideal of S is finite. In particular, S is periodic. Since RS is semisimple, [10, Thm. 2.4] tells us that S is torsion-disjoint with R. This completes our proof. ■

PROPOSITION 6. *If a commutative ring is a sum of several subrings with identities, then it is a ring with identity.*

Proof. Let R be a sum of R_i, $i = 1, \dots, n$, where R_i has identity 1_i. Put

$$e = \sum_{i=1}^{n} 1_i - \sum_{1 \le i < j \le n} 1_i 1_j + \sum_{1 \le i < j < k \le n} 1_i 1_j 1_k + \cdots + (-1)^{n-1} 1_1 1_2 \cdots 1_n.$$

It is routine to verify that

$$e = 1_i + (1 - 1_i) \left\{ \sum_{j \ne i} 1_j - \sum_{j<k;\, j,k \ne i} 1_j 1_k + \cdots + (-1)^{n-1} 1_1 1_2 \cdots 1_{i-1} 1_{i+1} \cdots 1_n \right\}.$$

Therefore e is an idempotent such that $e1_i = 1_i$ for all i. Given that $R = \sum_{i=1}^{n} R_i$, follows that e is the identity of R. ■

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